

$$3.1 \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$(x, y) \mapsto x^2 y$$

$$a) \frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \cdot 0 - 0}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

$$b) \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \lim_{h \rightarrow 0} \frac{(x_0+h)^2 y_0 - x_0^2 y_0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x_0^2 + 2x_0 h + h^2) y_0 - x_0^2 y_0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2x_0 y_0 h + h^2 y_0}{h} = \lim_{h \rightarrow 0} 2x_0 y_0 + h y_0$$

$$= 2x_0 y_0 \quad \checkmark$$

$$c) \frac{\partial f}{\partial y}(1, 2) = \lim_{h \rightarrow 0} \frac{f(1, 2+h) - f(1, 2)}{h} = \lim_{h \rightarrow 0} \frac{2+h-2}{h} = 1$$

$$d) \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x_0^2 (y_0+h) - x_0^2 y_0}{h} = \lim_{h \rightarrow 0} \frac{x_0^2 h}{h} = x_0^2$$

3.2

$$a) f(x, y) = 3x^2 + 2y^2$$

$$\frac{\partial f}{\partial x}(x, y) = 6x$$

$$\frac{\partial f}{\partial y}(x, y) = 4y$$

$$b) f(x, y) = \ln(x^2 - 3xy)$$

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}(\ln(x^2 - 3xy)) = \frac{1}{\ln(x^2 - 3xy)}(2x - 3y)$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}(\ln(x^2 - 3xy)) = -3x \left(\frac{1}{\ln(x^2 - 3xy)} \right)$$

$$c) f(x, y) = x^2 y^2 e^{2xy}$$

$$\frac{\partial f}{\partial x}(x, y) = y^2 x (2x e^{2xy} + 2x^2 y e^{2xy}) = 2x^2 y^2 e^{2xy} + 2x^3 y e^{2xy}$$

$$\frac{\partial f}{\partial y}(x, y) = x^2 (2y e^{2xy} + 2x^2 e^{2xy}) = 2x^2 y e^{2xy} + 2x^4 e^{2xy}$$

$$\frac{\partial}{\partial y}(x, y) = x \times (2y e^{-xy} + 2y^2 x e^{-xy}) = 2x^2 y e^{-xy} + 2y^2 x e^{-xy}$$

$$d) f(x, y) = e^x \ln(xy)$$

$$\frac{\partial f}{\partial x}(x, y) = e^x \ln(xy) + e^x \left(\frac{1}{x} \right)$$

$$\frac{\partial f}{\partial y}(x, y) = e^x \left(\frac{1}{y} \right)$$

$$\begin{aligned} (y^{\frac{1}{2}})' &= \frac{1}{2} y^{-\frac{1}{2}} \\ &= \frac{1}{2} \times \frac{1}{\sqrt{y}} \end{aligned}$$

$$e) f(x, y) = e^{\sin(x\sqrt{y})}$$

$$\frac{\partial f}{\partial x}(x, y) = \sqrt{y} \times \cos(x\sqrt{y}) \times e^{\sin(x\sqrt{y})}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{x}{2\sqrt{y}} \times \cos(x\sqrt{y}) \times e^{\sin(x\sqrt{y})}$$

$$f) f(x, y) = \frac{x^2 + y^2}{x^2 - y^2}$$

$$\frac{\partial f}{\partial x}(x, y) = \frac{2x(x^2 + y^2) - 2x(x^2 - y^2)}{(x^2 - y^2)^2}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{2y(x^2 - y^2) + 2y(x^2 + y^2)}{(x^2 - y^2)^2}$$

$$g) f(x, y) = x \cos(x) \cos(y)$$

$$\frac{\partial f}{\partial x}(x, y) = \cos y \times (\cos x \cdot (-x \sin x)) = -x \cos x \cos y \sin x$$

$$\frac{\partial f}{\partial y}(x, y) = -x \cos x \sin y$$

$$i) f(x, y) = x + y^2 x + \ln(\sin(x^2 + y))$$

$$\frac{\partial f}{\partial x}(x, y) = 1 + y^2 + 2x \cos(x^2 + y) \times \frac{1}{\sin(x^2 + y)}$$

$$\frac{\partial f}{\partial y}(x, y) = 2xy + \cos(x^2 + y) \times \frac{1}{\sin(x^2 + y)}$$

$$j) f(x, y, z) = z e^{x^2 + y^2}$$

$$\frac{\partial f}{\partial x}(x, y, z) = 2zx e^{x^2 + y^2}$$

$$\frac{\partial f}{\partial y}(x, y, z) = 2yz e^{x^2 + y^2}$$

$$\frac{\partial f}{\partial z}(x, y, z) = e^{x^2 + y^2}$$

$$k) f(x, y, z) = \ln(e^x + e^y)$$

$$\frac{\partial f}{\partial x}(x, y, z) = e^x \cdot \frac{1}{e^x + e^y}$$

$$\frac{\partial f}{\partial x}(x, y, z) = \frac{y}{e^x + z^4}$$

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{y}{e^x + z^4} \ln z \cdot \frac{1}{e^x + z^4}$$

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{y z^3}{z} \cdot \frac{1}{e^x + z^4}$$

$$1) f(x, y, z) = \frac{xy^3 + e^z}{x^3y - e^z}$$

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y, z) &= \frac{y^3(x^3y - e^z) - (xy^3 + e^z)(3yx^2)}{(x^3y - e^z)^2} \\ &= \frac{x^3y^4 - e^zy^3 - 3x^3y^4 - 3x^2ye^z}{(x^3y - e^z)^2} \\ &= \frac{-2x^3y^4 - 3x^2ye^z - e^zy^3}{(x^3y - e^z)^2} = \frac{y(-2x^3y^3 - 3x^2e^z - e^zy^2)}{(x^3y - e^z)^2} \end{aligned}$$

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{3xy^2(x^3y - e^z) - (xy^3 + e^z)(x^3)}{(x^3y - e^z)^2}$$

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{e^z(x^3y - e^z) - (xy^3 + e^z)(-e^z)}{(x^3y - e^z)^2}$$

3.3 $f_x(0,0), f_y(0,0)$???

$$a) f(x, y) = \begin{cases} \frac{x+y}{x^2+y^2} & (x, y) \neq (0,0) \\ \pi & (x, y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{h} - \pi}{h} = \lim_{h \rightarrow 0} \frac{1 - \pi h}{h^2} = +\infty$$

Não existe

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial x}(y, x) \Rightarrow \frac{\partial f}{\partial y}(0,0) = \frac{\partial f}{\partial x}(0,0)$$

$$b) f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2 + y^2} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

$$f_x(0,0) = f_y(0,0) = 0 \quad \checkmark$$

3.4)

$$a) \text{ se } f(x, y) = e^{xy}, \text{ então } x \frac{\partial f}{\partial x} = y \frac{\partial f}{\partial y}$$

$$\begin{aligned} \frac{\partial f}{\partial x}(1,2) &= 3 \\ \frac{\partial f}{\partial y}(1,2) &= \frac{\partial f}{\partial x}(2,1) = ? \\ \frac{\partial f}{\partial y}(2,1) &= \frac{\partial f}{\partial x}(1,2) = 3 \quad \checkmark \end{aligned}$$

$$2x + 2y$$

$$x \frac{\partial f}{\partial x} = x(y e^{xy})$$

$$y \frac{\partial f}{\partial y}(x, y) = y(x e^{xy}) = x(y e^{xy}) = x \frac{\partial f}{\partial x}(x, y)$$

$$b) \text{ se } f(x, y) = \ln(x^2 + y^2 + xy), \text{ entonces } x f_x + y f_y = 2$$

$$x f_x(x, y) = x(2x + y) \cdot \frac{1}{(x^2 + y^2 + xy)}$$

$$y f_y(x, y) = y(2y + x) \cdot \frac{1}{x^2 + y^2 + xy}$$

$$\begin{aligned} x f_x + y f_y &= \frac{x(2x + y) + y(2y + x)}{(x^2 + y^2 + xy)} = \frac{2x^2 + xy + 2y^2 + xy}{(x^2 + y^2 + xy)} \\ &= \frac{2(x^2 + y^2 + xy)}{(x^2 + y^2 + xy)} \\ &= 2 \end{aligned}$$

$$c) \text{ se } f(x, y, z) = x + \frac{x-y}{y-z}, \text{ entonces } f_x + f_y + f_z = 1$$

$$f_x(x, y, z) = 1 + \frac{(y-z)}{(y-z)^2} = 1 + \frac{1}{y-z}$$

$$f_y(x, y, z) = \frac{(-1)(y-z) - (x-y)}{(y-z)^2} = \frac{z-x}{(y-z)^2}$$

$$f_z(x, y, z) = -\frac{-1(x-y)}{(y-z)^2} = \frac{x-y}{(y-z)^2}$$

$$\begin{aligned} f_x + f_y + f_z &= 1 + \frac{1}{y-z} + \frac{z-x}{(y-z)^2} + \frac{x-y}{(y-z)^2} \\ &= 1 + \frac{1}{y-z} + \frac{z-y}{(y-z)^2} \\ &= 1 + \frac{1}{y-z} - \frac{1}{y-z} \\ &= 1 \end{aligned}$$

3.6

$$a) f(x, y) = x^2 y + x \quad P = (1, 0), \quad v = (1, 1)$$

$$\frac{\partial f}{\partial x}(P); f_x(P) \leftarrow \text{Derivada}$$

$$\frac{\partial f}{\partial v}(P) = D_f(P; \frac{v}{\|v\|}) \rightarrow \text{Derivada direccional}$$

$$D_f(P; v)$$

$$\|v\| = \|(1, 1)\| = \sqrt{2}$$

$$\frac{v}{\|v\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\|v\| = \sqrt{v_1^2 + v_2^2}$$

$$\begin{aligned} \frac{\partial f}{\partial v}(p) &= Df\left((1,0); \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) = \lim_{h \rightarrow 0} \frac{f\left((1,0) + h\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) - f(1,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right) - f(1,0)}{h} = \lim_{h \rightarrow 0} \frac{\left(1 + \frac{h}{\sqrt{2}}\right)^2 \frac{h}{\sqrt{2}} + 1 + \frac{h}{\sqrt{2}} - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{h}{\sqrt{2}} + \frac{2h^2}{2} + \frac{h^3}{2\sqrt{2}} + \frac{h}{\sqrt{2}}}{h} = \lim_{h \rightarrow 0} \frac{h\left(\frac{2}{\sqrt{2}} + h + \frac{h^2}{2\sqrt{2}}\right)}{h} = \frac{2}{\sqrt{2}} = \sqrt{2} \end{aligned}$$

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alternativamente temos:

Como f é derivável em $(1,0)$

$$Df(1,0); \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \underbrace{Df(1,0)}_L \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\begin{aligned} \nabla f(1,0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= (1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \\ \downarrow \\ \left(\frac{\partial f}{\partial x}(1,0), \frac{\partial f}{\partial y}(1,0)\right) \end{aligned}$$

$$\frac{\partial f}{\partial x}(x,y) = 2xy + 1, \quad \frac{\partial f}{\partial x}(1,0) = 1; \quad \frac{\partial f}{\partial y}(x,y) = x^2 \rightarrow \frac{\partial f}{\partial y}(1,0) = 1$$

$$b) f(x,y) = x^2 \sin(2y), \quad p = \left(1, \frac{\pi}{2}\right) \text{ e } v = (3, -4)$$

$$\left. \begin{aligned} f_x(x,y) &= 2x \sin(2y) \\ f_y(x,y) &= 2x^2 \cos(2y) \end{aligned} \right\} \rightarrow \text{Como os derivados parciais existem} \\ \text{e são contínuos então a função é derivável} \\ \text{no seu domínio}$$

$$f_x\left(1, \frac{\pi}{2}\right) = 0$$

$$f_y\left(1, \frac{\pi}{2}\right) = 2 \cos \pi = -2$$

$$\nabla f\left(1, \frac{\pi}{2}\right) = (0, -2)$$

$$\|v\| = \|(3, -4)\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

$$u = \frac{v}{\|v\|} = \left(\frac{3}{5}, -\frac{4}{5}\right)$$

$$Df(p; u) = Df\left(\left(1, \frac{\pi}{2}\right); \left(\frac{3}{5}, -\frac{4}{5}\right)\right) = Df\left(1, \frac{\pi}{2}\right) \left(\frac{3}{5}, -\frac{4}{5}\right)$$

$$= \nabla f\left(1, \frac{\pi}{2}\right) \cdot \left(\frac{3}{5}, -\frac{4}{5}\right) = (0, -2) \cdot \left(\frac{3}{5}, -\frac{4}{5}\right) = \frac{8}{5}$$

$$c) f(x,y,z) = x^2 + y^2 + z^2, \quad p = (1, 2, 3) \text{ e } v = (1, 1, 1)$$

$$f_x(x,y,z) = 2x \rightarrow f_x(1,2,3) = 2$$

$$f_y(x,y,z) = 2y \rightarrow f_y(1,2,3) = 4$$

$$f_2(x, y, z) = 2z \rightarrow f_2(1, 2, 3) = 6$$

$$\nabla f(1, 2, 3) = (2, 4, 6)$$

$$\|u\| = \sqrt{3}$$

$$u = \frac{u}{\|u\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$Df((1, 2, 3); \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)) = \nabla f(1, 2, 3) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$= (2, 4, 6) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{6}{\sqrt{3}}$$

$$= \frac{12}{\sqrt{3}} = 4 \cdot \frac{3}{\sqrt{3}} = 4\sqrt{3}$$

3.6

$$a) f(x, y) = x e^{-x+y}$$

$$f_x(x, y) = e^{-x+y} + (-1)x e^{-x+y} = e^{-x+y} - x e^{-x+y}$$

$$f_y(x, y) = x e^{-x+y}$$

$$\nabla f(x, y) = (e^{-x+y} - x e^{-x+y}, x e^{-x+y})$$

$$b) f(r, \theta) = r \sin \theta$$

$$f_r(r, \theta) = \sin \theta$$

$$f_\theta(r, \theta) = r \cos \theta$$

$$\nabla f(r, \theta) = (\sin \theta, r \cos \theta)$$

$$c) f(r, h) = \pi r^2 h$$

$$f_r(r, h) = 2\pi h r$$

$$f_h(r, h) = 2\pi r$$

$$\nabla f(r, h) = (2\pi h r, 2\pi r)$$

$$d) f(x, y, z) = x e^{x^2-y^2-z^2}$$

$$\frac{\partial f}{\partial x}(x, y, z) = e^{-x^2-y^2-z^2} - 2x^2 e^{-x^2-y^2-z^2} = (1-2x^2) e^{-x^2-y^2-z^2}$$

$$\frac{\partial f}{\partial y} = -2xy e^{-x^2-y^2-z^2}$$

$$\frac{\partial f}{\partial z} = -2xz e^{-x^2-y^2-z^2}$$

$$\nabla f(x, y, z) = ((1-2x^2) e^{-x^2-y^2-z^2}, -2xy e^{-x^2-y^2-z^2}, -2xz e^{-x^2-y^2-z^2})$$

$$= e^{-x^2-y^2-z^2} (1-2x^2, -2xy, -2xz)$$

$$e) f(x, y, z) = x y z$$

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + 1}$$

$$f_x(x, y, z) = \frac{yz(x^2 + y^2 + z^2 + 1) - 2x(xy z)}{(x^2 + y^2 + z^2 + 1)^2} = \frac{-x^2 y z + y z(y^2 + z^2 + 1)}{(x^2 + y^2 + z^2 + 1)^2}$$

$$f_y(x, y, z) = \frac{xz(x^2 + y^2 + z^2 + 1) - 2y(xy z)}{(x^2 + y^2 + z^2 + 1)^2} = \frac{-x y^2 z + x z(x^2 + z^2 + 1)}{(x^2 + y^2 + z^2 + 1)^2}$$

$$f_z(x, y, z) = \frac{xy(x^2 + y^2 + z^2 + 1) - 2z(xy z)}{(x^2 + y^2 + z^2 + 1)^2} = \frac{-x y z^2 + xy(x^2 + y^2 + 1)}{(x^2 + y^2 + z^2 + 1)^2}$$

$$\nabla f(x, y, z) = \frac{1}{(x^2 + y^2 + z^2 + 1)} (-x^2 y z + y z(y^2 + z^2 + 1), -x y^2 z + x z(x^2 + z^2 + 1), -x y z^2 + xy(x^2 + y^2 + 1))$$

$$f) f(x, y, z) = z^2 e^x \cos y$$

$$f_x(x, y, z) = z^2 e^x \cos y$$

$$f_y(x, y, z) = -z^2 e^x \sin y$$

$$f_z(x, y, z) = 2z e^x \cos y$$

$$\nabla f(x, y, z) = e^x (z^2 \cos y, -z^2 \sin y, 2z \cos y)$$

$$3.7 f(x, y) = x^2 + y^3, \quad p = (3, 1)$$

$$(a, b) \hookrightarrow (a, b, f(a, b))$$

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$= f(a, b) + \nabla f(a, b) \cdot (x - a, y - b)$$

$$= f(3, 1) + \nabla f(3, 1) \cdot (3 - 1, 1 - 1)$$

$$= 10 + (6, 3) \cdot (x - 3, y - 1)$$

$$= 10 + 6x - 18 + 3y - 3$$

$$= -11 + 6x + 3y$$