

4.1)

a) $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ tal que $f(x, y) = (x, y)$;

$$f(x, y) = (x, y)$$

$\downarrow$
 f_1

$\downarrow$
 f_2

$$\frac{\partial f_1}{\partial x} = 1 ; \quad \frac{\partial f_1}{\partial y} = 0$$

$$\frac{\partial f_2}{\partial x} = 0 ; \quad \frac{\partial f_2}{\partial y} = 1$$

$$J_f(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ tal que $f(x, y) = (xe^y + \cos y, x, x + e^y)$

$$\frac{\partial f_1}{\partial x} = e^y ; \quad \frac{\partial f_1}{\partial y} = xe^y - \sin y$$

$$\frac{\partial f_2}{\partial x} = 1 ; \quad \frac{\partial f_2}{\partial y} = 0$$

$$J_f(x, y) = \begin{pmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{pmatrix}$$

$$\frac{\partial f_3}{\partial x} = 1 ; \quad \frac{\partial f_3}{\partial y} = e^y$$

c) $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ tal que $f(x, y) = (xye^{xy}, x \sin y, 5xy^2)$

$$\begin{aligned} \frac{\partial}{\partial x}(xye^{xy}) &= ye^{xy} + xy(ye^{xy}) & \frac{\partial}{\partial y}(xye^{xy}) &= xe^{xy} + xy(xe^{xy}) \\ &= ye^{xy} + x y^2 e^{xy} & &= xe^{xy} + x^2 y e^{xy} \\ &= ye^{xy}(1 + xy) & &= xe^{xy}(1 + xy) \end{aligned}$$

$$\frac{\partial}{\partial x}(x \sin y) = \sin y ; \quad \frac{\partial}{\partial y}(x \sin y) = x \cos y$$

$$\frac{\partial}{\partial x}(5xy^2) = 5y^2 ; \quad \frac{\partial}{\partial y}(5xy^2) = 10xy$$

$$J_f(x, y) = \begin{pmatrix} ye^{xy}(1+xy) & xe^{xy}(1+xy) \\ \sin y & x \cos y \\ 5y^2 & 10xy \end{pmatrix}$$

d) $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ tal que $f(x, y, z) = (x - y, y + z)$

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$$\frac{\partial}{\partial x}(x-y) = 1 ; \frac{\partial}{\partial y}(x-y) = -1 ; \frac{\partial}{\partial z}(x-y) = 0$$

$$\frac{\partial}{\partial x}(y+z) = 0 ; \frac{\partial}{\partial y}(y+z) = 1 ; \frac{\partial}{\partial z}(y+z) = 1$$

$$J_f(x, y, z) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

c) $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ tal que $f(x, y, z) = (\overbrace{x+y+e^z}^{f_1}, x^2y)$

$$\frac{\partial f_1}{\partial x}(x, y, z) = 1 ; \frac{\partial}{\partial y}(x+y+e^z) = 1 ; \frac{\partial}{\partial z}(x+y+e^z) = e^z$$

$$\frac{\partial}{\partial x}(x^2y) = 2xy ; \frac{\partial}{\partial y}(x^2y) = x^2 ; \frac{\partial}{\partial z}(x^2y) = 0$$

$$J_f(x, y, z) = \begin{pmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{pmatrix}$$

4.2)

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$(x, y, z) \mapsto (x-y+z, x^2yz, xyz)$$

$$g: \mathbb{R}^3 \rightarrow \mathbb{R}^4$$

$$(x, y, z) \mapsto (xy, yz, 2x, xyz)$$

a) $Df((-1, 0, -1); (2, 3, -1))$

$$p = (-1, 0, -1); v = (2, 3, -1)$$

$$\frac{\partial}{\partial x}(x-y+z) = 1 ; \frac{\partial}{\partial y}(x-y+z) = -1 ; \frac{\partial}{\partial z}(x-y+z) = 1$$

$$\frac{\partial}{\partial x}(x^2yz) = 2xyz ; \frac{\partial}{\partial y}(x^2yz) = x^2z ; \frac{\partial}{\partial z}(x^2yz) = x^2y$$

$$\frac{\partial}{\partial x}(xyz) = yz ; \frac{\partial}{\partial y}(xyz) = xz ; \frac{\partial}{\partial z}(xyz) = xy$$

$$J_f(x, y, z) = \begin{pmatrix} 1 & -1 & 1 \\ xyz & x^2z & x^2y \\ yz & xz & xy \end{pmatrix}$$

$$J_f(-1, 0, -1) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Df(p; v) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{3 \times 3} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}_{3 \times 1} = \begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix}$$

$$\underline{\underline{Df((-1, 0, -1); (2, 3, -1)) = (-2, -3, 3)}}$$

$$Dg((-1, 0, -1); (2, 3, -1))$$

$$P = (-1, 0, -1)$$

$$u = (2, 3, -1)$$

$$\frac{\partial}{\partial x}(xy) = y; \frac{\partial}{\partial y}(xy) = x; \frac{\partial}{\partial z}(xy) = z$$

$$\frac{\partial}{\partial x}(yz) = 0; \frac{\partial}{\partial y}(yz) = z; \frac{\partial}{\partial z}(yz) = y$$

$$\frac{\partial}{\partial x}(zx) = z; \frac{\partial}{\partial y}(zx) = 0; \frac{\partial}{\partial z}(zx) = x$$

$$\frac{\partial}{\partial x}(xyz) = yz; \frac{\partial}{\partial y}(xyz) = xz; \frac{\partial}{\partial z}(xyz) = xy$$

$$J_g(x, y, z) = \begin{pmatrix} y & x & z \\ 0 & 0 & y \\ z & xz & xy \end{pmatrix}$$

$$J_g(-1, 0, -1) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$J_g(P; u) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}_{4 \times 3} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}_{3 \times 1} = \begin{pmatrix} -3 \\ -3 \\ 4 \\ 3 \end{pmatrix}$$

$$Jg((-1, 0, -1); (2, 3, -1)) = (-3, -3, 4, 3)$$

2)

$$Jf(-1, 0, 1)$$

$$Jf(x, y, z) = \begin{pmatrix} 1 & -1 & 1 \\ 2xyz & x^2z & x^2y \\ yz & xz & xy \end{pmatrix}$$

$$Df(-1, 0, 1) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$Dg(-1, 0, -1)$$

$$Jg(x, y, z) = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & 0 \\ yz & xz & xy \end{pmatrix}$$

$$/ 0 \quad -1 \quad 0 \backslash$$

$$Jg(-1, 0, -1) = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1.3)

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (3x, x + 2y)$$

1)

$$J_f(x, y, z) = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

2)

A função f é derivável em todo o domínio porque as suas componentes são polinómios de grau 1 (funções estritamente lineares). As suas derivadas parciais existem e são constantes (logo, perfeitamente contínuas) em todo o domínio. Pelo teorema da diferenciabilidade, como as derivadas parciais são contínuas, a função é diferenciável.

$$\Rightarrow Df(1, 2) = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

$$Df(1, 2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ x + 2y \end{pmatrix} = f(x, y)$$

$$b) Df(x_0, y_0) = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

1.4)

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$(x, y) \mapsto (2x^2, 3y, 2xy)$$

1)

$$J_f(x, y) = \begin{pmatrix} 4x & 0 \\ 0 & 3 \\ 2y & 2x \end{pmatrix}$$

2)

Todas as derivadas parciais existem e são contínuas pelo que a função f é de facto diferenciável

$$Df(1, 1) = \begin{pmatrix} 4 & 0 \\ 0 & 3 \\ 2 & 2 \end{pmatrix}$$

c)

$$Df(1, 1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix}$$

4.5)

$$1) f(x, y) = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial x}(x, y) = \frac{2y(x^2 + y^2)^2 - 2xy(2(2x)(x^2 + y^2))}{(x^2 + y^2)^4} = \frac{2y(x^2 + y^2)^2 - 8x^2y(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{2y(x^2 + y^2 - 4x^2)}{(x^2 + y^2)^3}$$

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{2x(x^2+y^2)^3 - 2xy(4y(x^2+y^2))}{(x^2+y^2)^4} = \frac{2x(x^2+y^2)^3 - 8x^2y}{(x^2+y^2)^4} = \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3} \\ &= \frac{2x^2y + 2y^3 - 8x^2y}{(x^2+y^2)^3} \\ &= \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}\end{aligned}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2x(x^2+y^2)^2 - 2xy(4y(x^2+y^2))}{(x^2+y^2)^4} = \frac{2x(x^2+y^2)^2 - 8xy^2}{(x^2+y^2)^4} = \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$\begin{aligned}\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{-12xy(x^2+y^2)^3 - (-6x^2y + 2y^3)(6x)(x^2+y^2)^2}{(x^2+y^2)^6} \\ &= \frac{-12xy(x^2+y^2)^3 - (-6x^2y + 2y^3)(6x)}{(x^2+y^2)^4} \\ &= \frac{-12x^3y - 12xy^3 - (-36x^3y + 12xy^3)}{(x^2+y^2)^4} \\ &= \frac{24x^3y - 24xy^3}{(x^2+y^2)^4} \\ &= \frac{24xy(x^2-y^2)}{(x^2+y^2)^4}\end{aligned}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{(-12xy(x^2+y^2)^3 - (2x^3 - 6xy^2)(6y)(x^2+y^2)^2)}{(x^2+y^2)^6}$$

$$\begin{aligned}&= [\dots] \\ &= \frac{24yx(y^2-x^2)}{(y^2+x^2)^4} \quad \textcircled{II}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(x,y) &= f_{xy}(x,y) = \frac{(6y^2 - 6x^2)(x^2+y^2)^3 - (2y^3 - 6x^2y)(6y)(x^2+y^2)^2}{(x^2+y^2)^6} \\ &= \frac{(6y^4 + 6x^2y^2 - 6x^2y^2 - 6x^4) - (12y^4 - 36x^2y^2)}{(x^2+y^2)^4}\end{aligned}$$

$$\dots + j)$$

$$= \frac{-6y^4 - 6x^4 + 36x^2y^2}{(x^2 + y^2)^4}$$

$$= \frac{6(6x^2y^2 - x^4 - y^4)}{(x^2 + y^2)^4}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = f_{yx}(x, y) = \frac{(6x^2 - 6y^2)(x^2 + y^2)^3 - (2x^3 - 6xy^2)(6x)(x^2 + y^2)^2}{(x^2 + y^2)^6}$$

$$= \frac{-6x^4 - 6y^4 + 36x^2y^2}{(x^2 + y^2)^4}$$

$$= \frac{6(6x^2y^2 - x^4 - y^4)}{(x^2 + y^2)^4}$$

$$1) f(x, y) = \cos(xy^2)$$

$$\frac{\partial f}{\partial x} = -y^2 \sin(xy^2)$$

$$\frac{\partial f}{\partial y} = -2xy \sin(xy^2)$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y}(x, y) &= f_{xy}(x, y) = -2y + (-y^2)(2xy \cos(xy^2)) \\ &= -2y - 2xy^3 \cos(xy^2) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x}(x, y) &= f_{yx}(x, y) = -2y + (-2xy)(y^2 \cos(xy^2)) \\ &= -2y - 2xy^3 \cos(xy^2) = f_{xy} \end{aligned}$$

Teorema di Schwarz

$$1) f(x, y) = e^{-xy^2} + y^3x^4$$

$$\frac{\partial f}{\partial x}(x, y) = -y^2 e^{-xy^2} + 4x^3y^3$$

$$\frac{\partial f}{\partial y}(x, y) = -2xy e^{-xy^2} + 3x^4y^2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = f_{xy}(x, y) = (-2y)(e^{-xy^2}) + (-y^2)(-2xy e^{-xy^2}) + 12x^3 y^2$$

$$= e^{-xy^2}(-2y + 2xy^3) + 12x^3 y^2$$

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = f_{yx}(x, y) = (-2y e^{-xy^2}) + (-2xy)(-y^2 e^{-xy^2}) + 12x^3 y^2$$

$$= e^{-xy^2}(-2y + 2xy^3) + 12x^3 y^2 = f_{xy} \quad \text{"(00)"}$$

8) $f_{xy} = f_{yx}$

1) $f_x(x, y) = 2x^3 \quad f_y = yx^2 + x$

$f_{xy} = 0$

$f_{yx} = 2xy + 1$

$0 \neq 2xy + 1$

2) $f_x(x, y) = x \cos y \quad f_y(x, y) = y \cos x$

$f_{xy} = x \cos y$

$f_{yx} = y \cos x$

$x \cos y \neq y \cos x$

1.9)

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{se } (x, y) \neq (0, 0) \\ 0 & \text{se } (x, y) = (0, 0) \end{cases}$$

1) Para $(x, y) \neq (0, 0)$

$$f'_x = \frac{y^3(x^2+y^2) - (xy^3)(2x)}{(x^2+y^2)^2} = \frac{-x^2y^3 + y^5}{(x^2+y^2)^2}$$

$$f'_y = \frac{3xy^2(x^2+y^2)^2 - (xy^3)(2y)}{(x^2+y^2)^2} = \frac{3x^3y^2 + xy^4}{(x^2+y^2)^2}$$

para $(x, y) = (0, 0)$

$$f'_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

2)

$$f_{xy}(0,0) = \lim_{h \rightarrow 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h^5 - (0^2)(h^3)}{(0^2 + h^2)^2} - 0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1$$

$$f_{yx}(0,0) = \lim_{h \rightarrow 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2h^3(0^2) + h(0^4)}{(h^2 + 0^2)^2} - 0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{0}{h^4} - 0}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h}$$

$$f_{xy}(0,0) = 1 \neq 0 = f_{yx}(0,0)$$

$$= 0 \quad \text{?}$$

que as derivadas parciais de ordem superior de esta função não são contínuas
origem

(11)