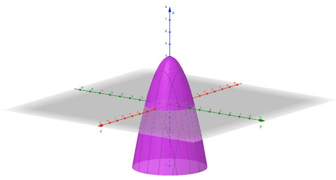
Folha 8

Friday, 22 May 2026



Exercício 8.2 Use integrais triplos para expressar o volume do sólido definido pela superfície de equação $z = a - x^2 - y^2$, com $a > 0$, e pelo plano XOY.

8.2)

$z = a - x^2 - y^2 \qquad a - x^2 - y^2 = 0$

$\Downarrow \qquad\qquad a - r^2 = 0 \Leftrightarrow r = \pm\sqrt{a}$

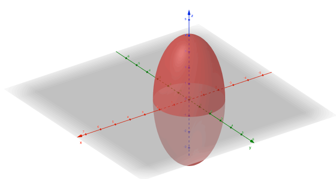
$z = a - r^2$

$$\cancel{\int_0^a \int_0^{2\pi} \int_0^{\sqrt{a}} r \, dr \, d\theta \, dz} = 2a\pi \cancel{\left[\frac{r^2}{2}\right]_0^{\sqrt{a}}} = 2a\pi \cancel{\frac{a}{2}} = a^2\pi$$

$$\int_0^{2\pi} \int_0^{\sqrt{a}} \int_0^{a-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{a}} rz \Big|_0^{a-r^2} dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{a}} ar - r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{ar^2}{2} - \frac{r^4}{4} \Big|_0^{\sqrt{a}} d\theta = \frac{1}{2} \int_0^{2\pi} a^2 - \frac{a^2}{2} \, d\theta = \pi \times \left(\frac{a^2}{2}\right) = \frac{a^2\pi}{2}$$

Exercício 8.3 Apresente um integral triplo que expresse o volume do elipsoide definido pela equação $4x^2 + 4y^2 + z^2 = 16$.



8.3) $4x^2 + 4y^2 + z^2 = 16 \Leftrightarrow z = \pm\sqrt{16 - 4x^2 - 4y^2}$

$z = 0 \Rightarrow 4x^2 + 4y^2 = 16 \Leftrightarrow x^2 + y^2 = 4$

$\Leftrightarrow y = \pm\sqrt{4 - x^2}$

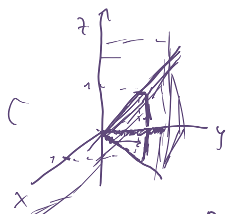
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{16-4x^2-4y^2}}^{\sqrt{16-4x^2-4y^2}} dz \, dy \, dx$$

$$\int_0^{2\pi} \int_0^{2} \int_{-\sqrt{16-4r^2}}^{\sqrt{16-4r^2}} r \, dz \, dr \, d\theta$$

Exercício 8.4 Faça um esboço da região de integração e reescreva o integral com ordem de integração $dx\,dy\,dz$:

a) $\int_0^1 \int_0^x \int_0^y f(x,y,z) \; dz\,dy\,dx$;

b) $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x,y,z) \; dz\,dy\,dx$.



a) $\int_0^1 \int_0^x \int_0^y f(x,y,z) \, dz \, dy \, dx$

$D = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1, \; 0 \leq y \leq x, \; 0 \leq z \leq y\}$

$\hookrightarrow \qquad 0 \leq z \leq y \leq x \leq 1$

$\Downarrow$

$0 \leq z \leq 1, \quad z \leq y \leq 1, \quad y \leq x \leq 1$

$$\int_0^1 \int_z^1 \int_y^1 f(x,y,z) \, dx \, dy \, dz$$

b) $$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z) \, dx \, dy \, dz$$

Exercício 8.5 Calcule, mudando eventualmente a ordem de integração,

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \int_1^3 \operatorname{sen} y^2 \; dz\,dy\,dx.$$

$$\left(\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \operatorname{sen} y^2 \, dy \, dx \right) 2$$

$\left(\int^{\sqrt{\frac{\pi}{2}}} \int^{y} \right)$

$$\left(\int_0^{\cdot 2}\int_0 \sin y^2 \, dx\, dy\right) 2$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} 2y \sin y^2 \, dy = \left(-\cos y^2 \Big|_0^{\sqrt{\frac{\pi}{2}}}\right) = 1$$

Exercício 8.6 Considerando $\mathcal{D} = \{(x,y,z) \in \mathbb{R}^3 : x, y > 0,\ \sqrt{x+y}+1 \leq z \leq 2\}$, calcule

$$\iiint_{\mathcal{D}} \frac{1}{\sqrt{xy}}\, d(x,y,z),$$

usando a mudança de variável definida por

$$\begin{array}{rccc} \Phi: & \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} & \longrightarrow & \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}. \\ & (u,v,w) & \longmapsto & (u^2, v^2, w) \end{array}$$

8.6 $D = \{(x,y,z) \in \mathbb{R}^3 : x, y > 0, \sqrt{x+y} + 1 \leq z \leq 2\}$

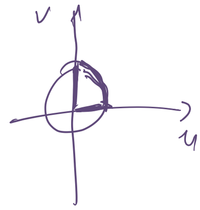
$$\iiint_D \frac{1}{\sqrt{xy}}\, d(x,y,z)$$

$\sqrt{x+y} + 1 = z$

$\hookrightarrow \sqrt{u^2+v^2} + 1 = w \rightsquigarrow w = 2 \Rightarrow \sqrt{u^2+v^2} + 1 = 2$

$\hookrightarrow \sqrt{u^2+v^2} = w - 1 \qquad \Rightarrow \sqrt{u^2+v^2} = 1$

$\hookrightarrow u^2+v^2 = (w-1)^2 \qquad \Rightarrow u^2+v^2 = 1$



$v = \sqrt{(w-1)^2 - u^2}$

$\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$

$(u,v,w) \to (u^2, v^2, w)$

$$\begin{cases} x = u^2 \\ y = v^2 \\ z = w \end{cases} \qquad J\Phi(u,v,w) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \begin{pmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2u\, 2v\, 1 = 4uv$$

$$\int_1^2 \int_0^{w-1} \int_0^{\sqrt{(w-1)^2-u^2}} \frac{1}{\cancel{uv}}\, 4\cancel{uv}\, dv\, du\, dw = \int_1^2 \int_0^{w-1} \int_0^{\sqrt{(w-1)^2-u^2}} 4\, dv\, du\, dw$$

$$\int_1^2 \int_0^{z-1} \int_0^{\frac{\pi}{2}} 4r\, d\theta\, dr\, dz = 4 \int_1^2 \int_0^{z-1} r\left(\frac{\pi}{2}\right) dr\, dz$$

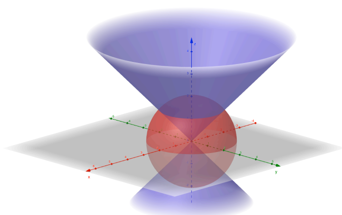
$$= 4 \int_1^2 \frac{\pi}{2} \times \left(\frac{r^2}{2}\right)\Big|_0^{z-1} dz$$

$$= 2\pi \int_1^2 \frac{(z-1)^2}{2}\, dz = \pi \left(\frac{z^3}{3} - \frac{2z^2}{2} + z \Big|_1^2\right)$$

$$= \pi \left(\frac{8}{3} - 4 + 2 - \frac{1}{3} + \cancel{1} - \cancel{1}\right)$$

$$= \pi\left(\frac{7}{3} - 2\right) = \frac{\pi}{3}$$

Exercício 8.12 Usando coordenadas esféricas, calcule o volume do sólido interior ao cone de equação $z^2 = x^2 + y^2$ e à esfera de equação $x^2 + y^2 + z^2 = 4$.



8.12 $z^2 = x^2 + y^2 \qquad x^2 + y^2 + z^2 = 4 \Rightarrow \underline{\rho = 2}$

$\Phi : \mathbb{R}_0^+ \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R}$

$$(\rho, \theta, \varphi) \longmapsto (\underbrace{\rho \sin\varphi \cos\theta}_{x}, \underbrace{\rho \sin\varphi \sin\theta}_{y}, \underbrace{\rho \cos\varphi}_{z})$$

$|\det J\Phi(\rho, \theta, \varphi)| = \rho^2 \sin\varphi$

$x^2 + y^2 = 4 - x^2 - y^2$

$2x^2 + 2y^2 = 4$

$x^2 + y^2 = 2$

$z^2 = x^2 + y^2 \Rightarrow \rho^2 \cos^2\varphi = \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta$

$\Rightarrow \rho^2 \cos^2\varphi = \rho^2 \sin^2\varphi (\cos^2\theta + \sin^2\theta)$

$\Rightarrow \rho^2 \cos^2\varphi = \rho^2 \sin^2\varphi$

$\Rightarrow 1 = \frac{\sin^2\varphi}{\quad} \Leftrightarrow \operatorname{tg}^2\varphi = 1 \Leftrightarrow \varphi = \frac{\pi}{\quad}$

$\cos^2 \varphi$ $\qquad$ 4

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3 \sin\varphi}{3}\Big|_0^2 \, d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} 8 \sin\varphi \, d\varphi \, d\theta = \frac{8}{3}\int_0^{2\pi} -\frac{\sqrt{2}}{2} + 1 \, d\theta = 2\pi\left(\frac{8}{3}\right)\left(\frac{2-\sqrt{2}}{2}\right) = \frac{16-8\sqrt{2}}{3}\pi$$